

# SEMICLASSICAL STATES FOR A STATIC SUPERCRITICAL KLEIN-GORDON-MAXWELL-PROCA SYSTEM ON A CLOSED RIEMANNIAN MANIFOLD

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**ABSTRACT.** We establish the existence of semiclassical states for a nonlinear Klein-Gordon-Maxwell-Proca system in static form, with Proca mass 1, on a closed Riemannian manifold.

Our results include manifolds of arbitrary dimension and allow supercritical nonlinearities. In particular, we exhibit a large class of 3-dimensional manifolds on which the system has semiclassical solutions for every exponent  $p \in (2, \infty)$ . The solutions we obtain concentrate at closed submanifolds of positive dimension as the singular perturbation parameter goes to zero.

## 1. INTRODUCTION

Let  $(\mathfrak{M}, \mathfrak{g})$  be a closed (i.e. compact and without boundary) smooth Riemannian manifold of dimension  $m \geq 2$ . Given real numbers  $\varepsilon > 0$ ,  $q > 0$ ,  $\omega \in \mathbb{R}$  and  $p \in (2, \infty)$ , and a real-valued  $\mathcal{C}^1$ -function  $\alpha$  such that  $\alpha(x) > \omega^2$  on  $\mathfrak{M}$ , we consider the system

$$(1.1) \quad \begin{cases} -\varepsilon^2 \Delta_{\mathfrak{g}} u + \alpha(x)u = u^{p-1} + \omega^2(q\mathfrak{v} - 1)^2 u & \text{on } \mathfrak{M}, \\ -\Delta_{\mathfrak{g}} \mathfrak{v} + (1 + q^2 u^2)\mathfrak{v} = qu^2 & \text{on } \mathfrak{M}, \\ u, \mathfrak{v} \in H_{\mathfrak{g}}^1(\mathfrak{M}), \quad u, \mathfrak{v} > 0. \end{cases}$$

The space  $H_{\mathfrak{g}}^1(\mathfrak{M})$  is the completion of  $\mathcal{C}^\infty(\mathfrak{M})$  with respect to the norm defined by  $\|v\|_{\mathfrak{g}}^2 := \int_{\mathfrak{M}} (|\nabla_{\mathfrak{g}} v|^2 + v^2) d\mu_{\mathfrak{g}}$ .

Solutions to this system correspond to standing waves of a Klein-Gordon-Maxwell-Proca (KGMP) system in static form (i.e. one in which the external Proca field is time-independent) with Proca mass 1.

KGMP-systems are massive versions of the more classical electrostatic Klein-Gordon-Maxwell (KGM) systems: KGM-systems are KGMP-systems with Proca mass 0, i.e. the second equation in (1.1) is replaced by

$$-\Delta_{\mathfrak{g}} \mathfrak{v} + q^2 u^2 \mathfrak{v} = qu^2.$$

Note that  $\mathfrak{v} = 1/q$  solves this last equation and reduces the KGM-system to a single Schrödinger equation in  $u$ . So for the system on a closed manifold the Proca

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formalism is more interesting and more appropriate. We refer to [11] for a detailed discussion on KGMP-systems and their physical meaning.

For  $\varepsilon = 1$  existence of solutions to system (1.1), which are stable with respect to the phase  $\omega$ , was established by Druet and Hebey [7] and Hebey and Truong [10] for manifolds of dimension  $m = 3$  and 4, and subcritical ( $2 < p < \frac{2m}{m-2}$ ) or critical ( $p = \frac{2m}{m-2}$ ) nonlinearities, under certain assumptions. For critical systems in dimension 3 Hebey and Wei [11] showed the existence of standing waves with multispikes amplitudes, which are unstable with respect to the phase, i.e. they blow up with  $k$  singularities as the phase  $\omega$  approaches some phase  $\omega_0$ .

Here we are interested in semiclassical states, i.e. in solutions to system (1.1) for  $\varepsilon$  small. The existence of semiclassical states for similar systems in flat domains  $\Omega$  in  $\mathbb{R}^m$  has been investigated e.g. in [4, 5, 15]. On closed 3-dimensional manifolds, the existence of semiclassical states to system (1.1), which concentrate at a single point as  $\varepsilon \rightarrow 0$ , was established in [8] and [9] for subcritical exponents  $p \in (2, 6)$ .

The results we present in this paper apply to manifolds of arbitrary dimension and include supercritical nonlinearities  $p > 2_m^*$ , where  $2_m^* := \frac{2m}{m-2}$  is the critical Sobolev exponent in dimension  $m \geq 3$  and  $2_2^* := \infty$ . In particular, we shall exhibit a large class of 3-dimensional manifolds on which the system (1.1) has semiclassical solutions for every exponent  $p \in (2, \infty)$ . The solutions  $u$  we obtain concentrate at closed submanifolds of  $\mathfrak{M}$  of positive dimension. Moreover, for fixed  $\varepsilon$ , they are stable with respect to the phase in the sense of [7].

Our approach consists in reducing system (1.1) to a system of a similar type on a manifold  $M$  of lower dimension but with the same exponent  $p$ . This way, if  $n := \dim M < \dim \mathfrak{M} =: m$  and  $p \in [2_m^*, 2_n^*)$ , then  $p$  is subcritical for the new system but it is critical or supercritical for the original one. Moreover, solutions of the new system which concentrate at a point in  $M$  as  $\varepsilon \rightarrow 0$  will give rise to solutions of the original system concentrating at a closed submanifold of  $\mathfrak{M}$  of dimension  $m - n$  as  $\varepsilon \rightarrow 0$ .

This approach was introduced by Ruf and Srikanth in [13], where a Hopf map is used to obtain the reduction. Reductions may also be performed by means of other maps which preserve the Laplace-Beltrami operator, or by considering warped products, or by a combination of both, see [3, 14] and the references therein. We describe these reductions in the following two subsections.

**1.1. Warped products.** If  $(M, g)$  and  $(N, h)$  are closed smooth Riemannian manifolds of dimensions  $n$  and  $k$  respectively, and  $f : M \rightarrow (0, \infty)$  is a  $\mathcal{C}^1$ -map, the *warped product*  $M \times_{f^2} N$  is the cartesian product  $M \times N$  equipped with the Riemannian metric  $\mathbf{g} := g + f^2 h$ .

For example, if  $M$  is a closed Riemannian submanifold of  $\mathbb{R}^\ell \times (0, \infty)$ , then

$$\mathfrak{M} := \{(y, z) \in \mathbb{R}^\ell \times \mathbb{R}^{k+1} : (y, |z|) \in M\},$$

with the induced euclidian metric, is isometric to the warped product  $M \times_{f^2} \mathbb{S}^k$ , where  $\mathbb{S}^k$  is the standard  $k$ -sphere and  $f(x_1, \dots, x_{\ell+1}) = x_{\ell+1}$ .

Let  $\pi_M : M \times_{f^2} N \rightarrow M$  be the projection. A straightforward computation gives the following result, cf. [6].

**Proposition 1.1.** *Let  $\beta : M \rightarrow \mathbb{R}$  and  $\alpha = \beta \circ \pi_M$ . Then  $u_\varepsilon, v_\varepsilon : M \rightarrow \mathbb{R}$  solve*

$$(1.2) \quad \begin{cases} -\varepsilon^2 \operatorname{div}_g (f^k(x) \nabla_g u) + f^k(x) \beta(x) u = f^k(x) u^{p-1} + \omega^2 f^k(x) (qv - 1)^2 u & \text{on } M, \\ -\operatorname{div}_g (f^k(x) \nabla_g v) + f^k(x) (1 + qu^2) v = q f^k(x) u^2 & \text{on } M, \end{cases}$$

iff  $u_\varepsilon := u_\varepsilon \circ \pi_M$ ,  $v_\varepsilon := v_\varepsilon \circ \pi_M : M \times_{f^2} N \rightarrow \mathbb{R}$  solve

$$(1.3) \quad \begin{cases} -\varepsilon^2 \Delta_g u + \alpha(x)u = u^{p-1} + \omega^2(qv - 1)^2 u & \text{on } M \times_{f^2} N, \\ -\Delta_g v + (1 + qu^2)v = qu^2 & \text{on } M \times_{f^2} N. \end{cases}$$

Note that the exponent  $p$  is the same for both systems. So if  $p \in (2_{n+k}^*, 2_n^*)$  then  $p$  is subcritical for (1.2) but supercritical for (1.3). Moreover, if the functions  $u_\varepsilon$  concentrate at a point  $\xi_0 \in M$  as  $\varepsilon \rightarrow 0$ , then the functions  $u_\varepsilon := u_\varepsilon \circ \pi_M$  concentrate at the submanifold  $\pi_M^{-1}(\xi_0) \cong (N, f^2(\xi_0)h)$  as  $\varepsilon \rightarrow 0$ .

**1.2. Harmonic morphisms.** Let  $(\mathfrak{M}, g)$  and  $(M, g)$  be closed Riemannian manifolds of dimensions  $m$  and  $n$  respectively. A *harmonic morphism* is a horizontally conformal submersion  $\pi : \mathfrak{M} \rightarrow M$  with dilation  $\lambda : \mathfrak{M} \rightarrow [0, \infty)$  which satisfies

$$(1.4) \quad (n - 2)\mathcal{H}(\nabla_g \ln \lambda) + (m - n)\kappa^\mathcal{V} = 0,$$

where  $\kappa^\mathcal{V}$  is the mean curvature of the fibers of  $\pi$  and  $\mathcal{H}$  is the projection of the tangent space of  $\mathfrak{M}$  onto the space orthogonal to the fibers, see [1].

So for  $n = 2$  a harmonic morphism is just a horizontally conformal submersion  $\pi : \mathfrak{M} \rightarrow M$  with minimal fibers. Typical examples are the Hopf fibration  $\mathbb{S}^3 \rightarrow \mathbb{S}^2$  whose fiber is  $\mathbb{S}^1$ , and the induced fibration  $\mathbb{R}P^3 \rightarrow \mathbb{S}^2$  with fiber  $\mathbb{R}P^1$ , see [1, Example 2.4.15]. They are, in fact, Riemannian submersions (i.e.  $\lambda \equiv 1$ ).

Harmonic morphisms preserve the Laplace-Beltrami operator, i.e.

$$\Delta_g(u \circ \pi) = \lambda^2 [(\Delta_g u) \circ \pi]$$

for every  $\mathcal{C}^2$ -function  $u : M \rightarrow \mathbb{R}$ . This fact yields the following result.

**Proposition 1.2.** *Assume there exist  $\beta : M \rightarrow \mathbb{R}$  and  $\mu : M \rightarrow (0, \infty)$  such that  $\beta \circ \pi = \alpha$  and  $\mu \circ \pi = \lambda^2$ . Then  $u_\varepsilon, v_\varepsilon : M \rightarrow \mathbb{R}$  solve the system*

$$(1.5) \quad \begin{cases} -\varepsilon^2 \Delta_g u + \frac{\beta(x)}{\mu(x)} u = \frac{1}{\mu(x)} u^{p-1} + \frac{\omega^2}{\mu(x)} (qv - 1)^2 u & \text{on } M, \\ -\Delta_g v + \frac{1}{\mu(x)} (1 + qu^2) v = \frac{q}{\mu(x)} u^2 & \text{on } M, \end{cases}$$

iff  $u_\varepsilon := u_\varepsilon \circ \pi_M$ ,  $v_\varepsilon := v_\varepsilon \circ \pi_M : \mathfrak{M} \rightarrow \mathbb{R}$  solve the system

$$(1.6) \quad \begin{cases} -\varepsilon^2 \Delta_g u + \alpha(x)u = u^{p-1} + \omega^2(qv - 1)^2 u & \text{on } \mathfrak{M}, \\ -\Delta_g v + (1 + qu^2)v = qu^2 & \text{on } \mathfrak{M}. \end{cases}$$

Again, if  $p \in (2_m^*, 2_n^*)$ , the system (1.5) is subcritical and the system (1.6) is supercritical and, if the functions  $u_\varepsilon$  concentrate at a point  $\xi_0 \in M$  as  $\varepsilon \rightarrow 0$ , the functions  $u_\varepsilon := u_\varepsilon \circ \pi_M$  concentrate at the  $(m - n)$ -dimensional submanifold  $\pi_M^{-1}(\xi_0)$  of  $\mathfrak{M}$  as  $\varepsilon \rightarrow 0$ .

**1.3. The main result for the general system.** Propositions 1.1 and 1.2 suggest studying a more general KGMP-system.

Let  $(M, g)$  be a closed Riemannian manifold of dimension  $n = 2$  or  $3$ ,  $a, b, c \in \mathcal{C}^1(M, \mathbb{R})$  be strictly positive functions,  $\varepsilon, q \in (0, \infty)$ ,  $p \in (2, 2_n^*)$ , and  $\omega \in \mathbb{R}$  be such that  $a(x) > \omega^2 b(x)$  on  $M$ . We consider the subcritical system

$$(1.7) \quad \begin{cases} -\varepsilon^2 \operatorname{div}_g (c(x) \nabla_g u) + a(x)u = b(x)u^{p-1} + b(x)\omega^2(qv - 1)^2 u & \text{in } M, \\ -\operatorname{div}_g (c(x) \nabla_g v) + b(x)(1 + q^2 u^2)v = b(x)qu^2 & \text{in } M, \\ u, v \in H_g^1(M), \quad u, v > 0. \end{cases}$$

**Theorem 1.3.** *Let  $K$  be a  $\mathcal{C}^1$ -stable critical set of the function  $\Gamma : M \rightarrow \mathbb{R}$  given by*

$$\Gamma(x) := \frac{c(x)^{\frac{n}{2}} a(x)^{\frac{p}{p-2} - \frac{n}{2}}}{b(x)^{\frac{2}{p-2}}}.$$

*Then, for  $\varepsilon$  small enough, the system (1.7) has a solution  $(u_\varepsilon, v_\varepsilon)$  such that  $u_\varepsilon$  concentrates at a point  $\xi_0 \in K$  as  $\varepsilon \rightarrow 0$ .*

Recall that  $K$  is a  $\mathcal{C}^1$ -stable critical set of a function  $f \in \mathcal{C}^1(M, \mathbb{R})$  if  $K \subset \{x \in M : \nabla_g f(x) = 0\}$  and for any  $\mu > 0$  there exists  $\delta > 0$  such that, if  $h \in \mathcal{C}^1(M, \mathbb{R})$  with

$$\max_{d_g(x, K) \leq \mu} |f(x) - h(x)| + |\nabla_g f(x) - \nabla_g h(x)| \leq \delta,$$

then  $h$  has a critical point  $x_0$  with  $d_g(x_0, K) \leq \mu$ . Here  $d_g$  denotes the geodesic distance associated to the Riemannian metric  $g$ .

**1.4. The main results for the KGMP-system.** Theorem 1.3, together with Propositions 1.1 and 1.2, yields the following results.

**Theorem 1.4.** *Let  $\mathfrak{M}$  be the warped product  $M \times_{f^2} N$  of two closed Riemannian manifolds  $(M, g)$  and  $(N, h)$  with  $n := \dim M = 2$  or  $3$ . Set  $k := \dim N$ , and let  $p \in (2, \infty)$  if  $n = 2$  and  $p \in (2, 6)$  if  $n = 3$ . Assume there exists  $\beta \in \mathcal{C}^1(M, \mathbb{R})$  such that  $\alpha = \beta \circ \pi_M$  and let  $K$  be a  $\mathcal{C}^1$ -stable critical set for the function  $\Gamma := f^k \beta^{\frac{p}{p-2} - \frac{n}{2}}$  on  $M$ . Then, for  $\varepsilon$  small enough, the KGMP-system (1.1) has a solution  $(\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon)$  such that  $\mathbf{u}_\varepsilon$  concentrates at the submanifold  $\pi_M^{-1}(\xi_0) \cong (N, f^2(\xi_0)h)$  for some  $\xi_0 \in K$  as  $\varepsilon \rightarrow 0$ .*

**Theorem 1.5.** *Assume there exist a closed Riemannian manifold  $M$  with  $n := \dim M = 2$  or  $3$  and a harmonic morphism  $\pi : \mathfrak{M} \rightarrow M$  whose dilation  $\lambda$  is such that  $\mu \circ \pi = \lambda^2$ . Assume further that  $\alpha = \beta \circ \pi$  with  $\beta \in \mathcal{C}^1(M, \mathbb{R})$ . Let  $p \in (2, \infty)$  if  $n = 2$  and  $p \in (2, 6)$  if  $n = 3$ , and let  $K$  be a  $\mathcal{C}^1$ -stable critical set for the function  $\Gamma := \beta^{\frac{p}{p-2} - \frac{n}{2}} \mu^{\frac{n}{2} - 1}$  on  $M$ . Then, for  $\varepsilon$  small enough, the KGMP-system (1.1) has a solution  $(\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon)$  such that  $\mathbf{u}_\varepsilon$  concentrates at the submanifold  $\pi^{-1}(\xi_0)$  of  $\mathfrak{M}$  for some  $\xi_0 \in K$  as  $\varepsilon \rightarrow 0$ .*

This last result applies, in particular, to the standard 3-sphere  $\mathfrak{M} = \mathbb{S}^3$  and the real projective space  $\mathfrak{M} = \mathbb{R}P^3$  for all  $p \in (2, \infty)$  with  $\mu = \lambda \equiv 1$ , see subsection 1.2.

The rest of the paper is devoted to the proof of Theorem 1.3. In section 2 we reduce the system to a single equation and give the outline of the proof of Theorem 1.3, which follows the well-known Lyapunov-Schmidt reduction procedure. In section 3 we establish the Lyapunov-Schmidt reduction and in section 4 we derive the expansion of the reduced energy functional. Section 5 is devoted to the proof of some technical results.

## 2. OUTLINE OF THE PROOF OF THEOREM 1.3

**2.1. Reduction to a single equation.** First, we reduce the system to a single equation. To overcome the problems caused by the competition between  $u$  and  $v$ , using an idea of Benci and Fortunato [2], we consider the map  $\Psi : H_g^1(M) \rightarrow H_g^1(M)$  defined by the equation

$$(2.1) \quad -\operatorname{div}_g(c(x)\nabla_g \Psi(u)) + b(x)(1 + q^2 u^2)\Psi(u) = b(x)qu^2.$$

It follows from standard variational arguments that  $\Psi$  is well-defined in  $H_g^1(M)$ .

Using the maximum principle and regularity theory it is not hard to prove that

$$(2.2) \quad 0 < \Psi(u) < 1/q \quad \text{for all } u \in H_g^1(M).$$

For the proofs of the following two lemmas we refer to [7].

**Lemma 2.1.** *The map  $\Psi : H_g^1(M) \rightarrow H_g^1(M)$  is of class  $C^1$ , and its differential  $V_u := \Psi'(u)$  at  $u$  is defined by*

$$(2.3) \quad -\operatorname{div}_g(c(x)\nabla_g V_u[h]) + b(x)(1 + q^2 u^2) V_u[h] = 2b(x)qu(1 - q\Psi(u))h$$

for every  $h \in H_g^1(M)$ . Moreover,

$$0 \leq \Psi'(u)[u] \leq \frac{2}{q} \quad \text{for all } u \in H_g^1(M).$$

**Lemma 2.2.** *The map  $\Theta : H_g^1(M) \rightarrow \mathbb{R}$  given by*

$$\Theta(u) := \frac{1}{2} \int_M b(x)(1 - q\Psi(u))u^2 d\mu_g$$

is of class  $C^1$  and

$$\Theta'(u)[h] = \int_M b(x)(1 - q\Psi(u))^2 u h d\mu_g \quad \text{for all } u, h \in H_g^1(M).$$

Next, we introduce the functionals  $I_\varepsilon, J_\varepsilon, G_\varepsilon : H_g^1(M) \rightarrow \mathbb{R}$  given by

$$(2.4) \quad I_\varepsilon(u) := J_\varepsilon(u) + \frac{\omega^2}{2} G_\varepsilon(u),$$

where

$$J_\varepsilon(u) := \frac{1}{2\varepsilon^2} \int_M [\varepsilon^2 c(x)|\nabla_g u|^2 + d(x)u^2] d\mu_g - \frac{1}{p\varepsilon^2} \int_M b(x)(u^+)^p d\mu_g$$

with  $d(x) := a(x) - \omega^2 b(x)$ , and

$$G_\varepsilon(u) := \frac{q}{\varepsilon^2} \int_M b(x)\Psi(u)u^2 d\mu_g.$$

From Lemma 2.2 we deduce that

$$\frac{1}{2} G'_\varepsilon(u)[\varphi] = \frac{1}{\varepsilon^2} \int_M b(x)[2q\Psi(u) - q^2\Psi^2(u)]u\varphi d\mu_g.$$

Hence,

$$I'_\varepsilon(u)\varphi = \frac{1}{\varepsilon^2} \int_M \varepsilon^2 c(x)\nabla_g u \nabla_g \varphi + a(x)u\varphi - b(x)(u^+)^{p-1}\varphi - b(x)\omega^2(1 - q\Psi(u))^2 u\varphi d\mu_g.$$

Therefore, if  $u$  is a critical point of the functional  $I_\varepsilon$ , then  $u$  solves the problem

$$(2.5) \quad \begin{cases} -\varepsilon^2 \operatorname{div}_g(c(x)\nabla_g u) + (a(x) - \omega^2 b(x))u + \omega^2 q b(x)\Psi(u)(2 - q\Psi(u))u = b(x)(u^+)^{p-1}, \\ u \in H_g^1(M). \end{cases}$$

If  $u \neq 0$  by the maximum principle and regularity theory we have that  $u > 0$ . Thus the pair  $(u, \Psi(u))$  is a solution of the system (1.7). This reduces the existence problem for the system (1.7) to showing that the functional  $I_\varepsilon$  has a nontrivial critical point.

**2.2. The limit problems.** Theorem 1.3 concerns manifolds of dimensions 2 and 3. To simplify the exposition we shall treat in full detail only the case  $n = 2$ . Everything can be extended in a straightforward way to the case  $n = 3$ , except for the estimates in section 5. These estimates, however, were computed in the appendix of [9] for  $n = 3$ .

Henceforth, we assume that  $\dim M = 2$ . We fix  $r > 0$  smaller than the injectivity radius of  $M$ . We identify the tangent space of  $M$  at  $\xi$  with  $\mathbb{R}^2$  and denote by  $B(x, r)$  the ball in  $\mathbb{R}^2$  centered at  $x$  of radius  $r$  and by  $B_g(\xi, r)$  the ball in  $M$  centered at  $\xi$  of radius  $r$ , with respect to the distance induced by the Riemannian metric  $g$ . The exponential map  $\exp_\xi : B(0, r) \rightarrow B_g(\xi, r)$  provides local coordinates on  $M$ , which are called normal coordinates. We denote by  $g_\xi$  the Riemannian metric at  $\xi$  given in normal coordinates by the matrix  $(g_{ij})$ . We denote the inverse matrix by  $(g^{ij}(z)) := (g_{ij}(z))^{-1}$  and write  $|g_\xi(z)| := \det(g_{ij}(z))$ . Then, we have that

$$(2.6) \quad g^{ij}(\varepsilon z) = \delta_{ij} + \frac{\varepsilon^2}{2} \sum_{r,k=1}^n \frac{\partial^2 g^{ij}}{\partial z_r \partial z_k}(0) z_r z_k + O(\varepsilon^3 |z|^3) = \delta_{ij} + o(\varepsilon),$$

$$(2.7) \quad |g(\varepsilon z)|^{\frac{1}{2}} = 1 - \frac{\varepsilon^2}{4} \sum_{i,r,k=1}^n \frac{\partial^2 g^{ii}}{\partial z_r \partial z_k}(0) z_r z_k + O(\varepsilon^3 |z|^3) = 1 + o(\varepsilon).$$

Here  $\delta_{ij}$  denotes the Kronecker symbol.

For  $p \in (2, \infty)$  and  $\xi \in M$ , set

$$A(\xi) := \frac{a(\xi)}{c(\xi)}, \quad B(\xi) := \frac{b(\xi)}{c(\xi)}, \quad \gamma(\xi) := \left( \frac{a(\xi)}{b(\xi)} \right)^{\frac{1}{p-2}}.$$

We consider the problem

$$-c(\xi)\Delta V + a(\xi)V = b(\xi)V^{p-1}, \quad V \in H^1(\mathbb{R}^2),$$

and denote by  $V^\xi$  its unique positive spherically symmetric solution. This problem is equivalent to

$$-\Delta V + A(\xi)V = B(\xi)V^{p-1}, \quad V \in H^1(\mathbb{R}^2).$$

The function  $V^\xi$  and its derivatives decay exponentially at infinity.  $V^\xi$  can be written as

$$V^\xi(z) = \gamma(\xi)U(\sqrt{A(\xi)}z),$$

where  $U$  is the unique positive spherically symmetric solution to

$$-\Delta U + U = U^{p-1}, \quad U \in H^1(\mathbb{R}^2).$$

For  $\xi \in M$  and  $\varepsilon > 0$  we define  $W_{\varepsilon,\xi} \in H_g^1(M)$  by

$$W_{\varepsilon,\xi}(x) := \begin{cases} V^\xi\left(\frac{1}{\varepsilon}\exp_\xi^{-1}(x)\right)\chi\left(\exp_\xi^{-1}(x)\right) & \text{if } x \in B_g(\xi, r), \\ 0 & \text{otherwise,} \end{cases}$$

where  $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$  is a radial cut-off function such that  $\chi(z) = 1$  if  $|z| \leq r/2$  and  $\chi(z) = 0$  if  $|z| \geq r$ . Setting  $V_\varepsilon(z) := V\left(\frac{z}{\varepsilon}\right)$  and  $y := \exp_\xi^{-1}x$  we have that

$$W_{\varepsilon,\xi}(\exp_\xi(y)) = V^\xi\left(\frac{y}{\varepsilon}\right)\chi(y) = V_\varepsilon^\xi(y)\chi(y),$$

so the function  $W_{\varepsilon,\xi}$  is simply the function  $V^\xi$  rescaled, cut off and read in normal coordinates at  $\xi$  in  $M$ .

Similarly, for  $i = 1, 2$  we define

$$Z_{\varepsilon, \xi}^i(x) = \begin{cases} \psi_{\xi}^i\left(\frac{1}{\varepsilon} \exp_{\xi}^{-1}(x)\right) \chi\left(\exp_{\xi}^{-1}(x)\right) & \text{if } x \in B_g(\xi, r), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\psi_{\xi}^i(\eta) = \frac{\partial}{\partial \eta_i} V^{\xi}(\eta) = \gamma(\xi) \sqrt{A(\xi)} \frac{\partial U}{\partial \eta_i}(\sqrt{A(\xi)} \eta).$$

The functions  $\psi_{\xi}^i$  are solutions of the linearized equation

$$-\Delta \psi + A(\xi) \psi = (p-1) B(\xi) (V^{\xi})^{p-2} \psi \quad \text{in } \mathbb{R}^2.$$

**Proposition 2.3.** *There is a positive constant  $C$  such that*

$$\langle Z_{\varepsilon, \xi}^h, Z_{\varepsilon, \xi}^k \rangle_{\varepsilon} = C \delta_{hk} + o(1),$$

as  $\varepsilon \rightarrow 0$ .

*Proof.* From the Taylor expansions of  $g^{ij}(\varepsilon z)$ ,  $|g(\varepsilon z)|^{\frac{1}{2}}$ ,  $a(\exp_{\xi}(\varepsilon z))$  and  $c(\exp_{\xi}(\varepsilon z))$  we obtain

$$\begin{aligned} \langle Z_{\varepsilon, \xi}^h, Z_{\varepsilon, \xi}^k \rangle_{\varepsilon} &= \frac{1}{\varepsilon^2} \int_M \varepsilon^2 c(x) \nabla_g Z_{\varepsilon, \xi}^h(x) \nabla_g Z_{\varepsilon, \xi}^k(x) + d(x) Z_{\varepsilon, \xi}^h(x) Z_{\varepsilon, \xi}^k(x) d\mu_g \\ &= \int_{B(0, r/\varepsilon)} \sum_{ij} c(\exp_{\xi}(\varepsilon z)) g_{\xi}^{ij}(\varepsilon z) \frac{\partial}{\partial z_i} (\psi_{\xi}^h(z) \chi(\varepsilon z)) \frac{\partial}{\partial z_j} (\psi_{\xi}^k(z) \chi(\varepsilon z)) |g_{\xi}(\varepsilon z)|^{\frac{1}{2}} dz \\ &\quad + \int_{B(0, r/\varepsilon)} d(\exp_{\xi}(\varepsilon z)) \psi_{\xi}^h(z) \psi_{\xi}^k(z) \chi^2(\varepsilon z) |g_{\xi}(\varepsilon z)|^{\frac{1}{2}} dz \\ &= c(\xi) \int_{\mathbb{R}^2} \nabla \psi_{\xi}^h \nabla \psi_{\xi}^k dz + d(\xi) \int_{\mathbb{R}^2} \psi_{\xi}^h \psi_{\xi}^k dz + o(1) = C \delta_{hk} + o(1), \end{aligned}$$

as claimed.  $\square$

Next, we compute the derivatives of  $W_{\varepsilon, \xi}$  with respect to  $\xi$  in normal coordinates. Fix  $\xi_0 \in M$ . We write the points  $\xi \in B_g(\xi_0, r)$  as

$$\xi = \xi(y) = \exp_{\xi_0}(y) \quad \text{with } y \in B(0, r).$$

We define

$$\mathcal{E}(y, x) = \exp_{\xi(y)}^{-1}(x) = \exp_{\exp_{\xi_0}(y)}^{-1}(x),$$

where  $x \in B_g(\xi(y), r)$  and  $y \in B(0, r)$ . Then we can write

$$\begin{aligned} W_{\varepsilon, \xi(y)}(x) &= \gamma(\xi(y)) U_{\varepsilon}(\sqrt{A(\xi(y))} \exp_{\xi(y)}^{-1}(x)) \chi(\exp_{\xi(y)}^{-1}(x)) \\ &= \tilde{\gamma}(y) U_{\varepsilon}(\sqrt{\tilde{A}(y)} \mathcal{E}(y, x)) \chi(\mathcal{E}(y, x)) \end{aligned}$$

where  $\tilde{A}(y) = A(\exp_{\xi_0}(y))$  and  $\tilde{\gamma}(y) = \gamma(\exp_{\xi_0}(y))$ . Thus we have

$$\begin{aligned} \frac{\partial}{\partial y_s} W_{\varepsilon, \xi(y)} \Big|_{y=0} &= \left( \frac{\partial}{\partial y_s} \tilde{\gamma}(y) \Big|_{y=0} \right) U \left( \frac{1}{\varepsilon} \sqrt{\tilde{A}(0)} \mathcal{E}(0, x) \right) \chi(\mathcal{E}(0, x)) \\ &\quad + \tilde{\gamma}(0) U \left( \frac{1}{\varepsilon} \sqrt{\tilde{A}(0)} \mathcal{E}(0, x) \right) \frac{\partial}{\partial y_s} \chi(\mathcal{E}(y, x)) \Big|_{y=0} \\ &\quad + \tilde{\gamma}(0) \chi(\mathcal{E}(0, x)) \frac{\partial}{\partial y_s} U \left( \frac{1}{\varepsilon} \sqrt{\tilde{A}(y)} \mathcal{E}(y, x) \right) \Big|_{y=0}. \end{aligned}$$

If  $x = \exp_{\xi_0} \varepsilon z$ ,  $\xi_0 = \xi(0)$ , then  $\mathcal{E}(0, x) = \varepsilon z$  and we have

$$(2.8) \quad \begin{aligned} \frac{\partial}{\partial y_s} W_{\varepsilon, \xi(y)} \Big|_{y=0} &= \left( \frac{\partial}{\partial y_s} \tilde{\gamma}(y) \Big|_{y=0} \right) U(\sqrt{\tilde{A}(0)} z) \chi(\varepsilon z) \\ &+ \tilde{\gamma}(0) U \left( \sqrt{\tilde{A}(0)} z \right) \frac{\partial \chi}{\partial \eta_k}(\varepsilon z) \frac{\partial}{\partial y_s} \mathcal{E}_k(y, \exp_{\xi_0} \varepsilon z) \Big|_{y=0} \\ &+ \tilde{\gamma}(0) \chi(\varepsilon z) \frac{\sqrt{\tilde{A}(0)}}{\varepsilon} \frac{\partial U}{\partial \eta_k} \left( \sqrt{\tilde{A}(0)} z \right) \frac{\partial}{\partial y_s} \mathcal{E}_k(y, \exp_{\xi_0} \varepsilon z) \Big|_{y=0}. \end{aligned}$$

We also recall the following Taylor expansions:

$$(2.9) \quad \frac{\partial}{\partial y_h} \mathcal{E}_k(0, \exp_{\xi_0} \varepsilon z) = -\delta_{hk} + O(\varepsilon^2 |z|^2).$$

**2.3. Outline of the proof of Theorem 1.3.** Let  $H_\varepsilon$  denote the Hilbert space  $H_g^1(M)$  equipped with the inner product

$$\langle u, v \rangle_\varepsilon := \frac{1}{\varepsilon^2} \left( \varepsilon^2 \int_M c(x) \nabla_g u \nabla_g v \, d\mu_g + \int_M d(x) uv \, d\mu_g \right),$$

which induces the norm

$$\|u\|_\varepsilon^2 := \frac{1}{\varepsilon^2} \left( \varepsilon^2 \int_M c(x) |\nabla_g u|^2 \, d\mu_g + \int_M d(x) u^2 \, d\mu_g \right),$$

with  $d(x) := a(x) - \omega^2 b(x) > 0$ . Similarly, let  $L_\varepsilon^q$  be the Banach space  $L_g^q(M)$  with the norm

$$|u|_{q, \varepsilon} := \left( \frac{1}{\varepsilon^2} \int_M |u|^q \, d\mu_g \right)^{1/q}.$$

Since we are assuming that  $\dim M = 2$ , for each  $q \geq 2$  the embedding  $H_\varepsilon \hookrightarrow L_\varepsilon^q$  is continuous. In fact, there is a positive constant  $C$ , independent of  $\varepsilon$ , such that

$$(2.10) \quad |u|_{q, \varepsilon} \leq C \|u\|_\varepsilon \quad \forall u \in H_\varepsilon,$$

Moreover, this embedding is compact.

Fix  $p \in (2, \infty)$ . The adjoint operator  $i_\varepsilon^* : L_\varepsilon^{p'} \rightarrow H_\varepsilon$ ,  $p' := \frac{p}{p-1}$ , to the embedding  $i_\varepsilon : H_\varepsilon \hookrightarrow L_\varepsilon^p$  is defined by

$$\begin{aligned} u = i_\varepsilon^*(v) &\Leftrightarrow \langle u, \varphi \rangle_\varepsilon = \frac{1}{\varepsilon^2} \int_M v \varphi \quad \forall \varphi \in H_\varepsilon \\ &\Leftrightarrow -\varepsilon^2 \operatorname{div}_g (c(x) \nabla_g u) + d(x) u = v, \quad u \in H_g^1(M). \end{aligned}$$

One has that

$$(2.11) \quad \|i_\varepsilon^*(v)\|_\varepsilon \leq C |v|_{p', \varepsilon} \quad \forall v \in L_\varepsilon^{p'},$$

where the constant  $C$  does not depend on  $\varepsilon$ .

Using the adjoint operator we can rewrite problem (2.5) as

$$(2.12) \quad u = i_\varepsilon^* [b(x) f(u) + \omega^2 b(x) g(u)], \quad u \in H_\varepsilon,$$

where

$$f(u) := (u^+)^{p-1} \quad \text{and} \quad g(u) := (q^2 \Psi^2(u) - 2q \Psi(u)) u.$$

Let

$$K_{\varepsilon, \xi} := \operatorname{Span} \{Z_{\varepsilon, \xi}^1, Z_{\varepsilon, \xi}^2\}$$



and

$$K_{\varepsilon,\xi}^\perp := \left\{ \phi \in H_\varepsilon : \langle \phi, Z_{\varepsilon,\xi}^i \rangle_\varepsilon = 0, \ i = 1, 2 \right\}.$$

We denote the projections onto these subspaces by

$$\Pi_{\varepsilon,\xi} : H_\varepsilon \rightarrow K_{\varepsilon,\xi} \quad \text{and} \quad \Pi_{\varepsilon,\xi}^\perp : H_\varepsilon \rightarrow K_{\varepsilon,\xi}^\perp.$$

We look for a solution of (2.5) of the form

$$u_\varepsilon := W_{\varepsilon,\xi} + \phi \quad \text{with} \quad \phi \in K_{\varepsilon,\xi}^\perp.$$

This is equivalent to solving the pair of equations

$$(2.13) \quad \Pi_{\varepsilon,\xi}^\perp \{ W_{\varepsilon,\xi} + \phi - i_\varepsilon^* [b(x)f(W_{\varepsilon,\xi} + \phi) + \omega^2 b(x)g(W_{\varepsilon,\xi} + \phi)] \} = 0,$$

$$(2.14) \quad \Pi_{\varepsilon,\xi} \{ W_{\varepsilon,\xi} + \phi - i_\varepsilon^* [b(x)f(W_{\varepsilon,\xi} + \phi) + \omega^2 b(x)g(W_{\varepsilon,\xi} + \phi)] \} = 0.$$

The first step of the proof of Theorem 1.3 is to solve equation (2.13). More precisely, for any fixed  $\xi \in M$  and  $\varepsilon$  small enough, we will show that there is a function  $\phi \in K_{\varepsilon,\xi}^\perp$  such that (2.13) holds. To do this we consider the linear operator  $L_{\varepsilon,\xi} : K_{\varepsilon,\xi}^\perp \rightarrow K_{\varepsilon,\xi}^\perp$  given by

$$L_{\varepsilon,\xi}(\phi) := \Pi_{\varepsilon,\xi}^\perp \{ \phi - i_\varepsilon^* [b(x)f'(W_{\varepsilon,\xi})\phi] \}.$$

For the proof of the following statement we refer to Lemma 4.1 of [3] (see also Proposition 3.1 of [12]).

**Proposition 2.4.** *There exist  $\varepsilon_0 > 0$  and  $C > 0$  such that, for every  $\varepsilon \in (0, \varepsilon_0)$ ,  $\xi \in M$  and  $\phi \in K_{\varepsilon,\xi}^\perp$ ,*

$$\|L_{\varepsilon,\xi}(\phi)\|_\varepsilon \geq C\|\phi\|_\varepsilon.$$

This result allows to use a contraction mapping argument to solve equation (2.13). The following statement is proved in section 3.

**Proposition 2.5.** *There exist  $\varepsilon_0 > 0$  and  $C > 0$  such that, for each  $\xi \in M$  and each  $\varepsilon \in (0, \varepsilon_0)$ , there exists a unique  $\phi_{\varepsilon,\xi} \in K_{\varepsilon,\xi}^\perp$  which solves equation (2.13). Moreover,*

$$\|\phi_{\varepsilon,\xi}\|_\varepsilon \leq C\varepsilon.$$

*The map  $\xi \mapsto \phi_{\varepsilon,\xi}$  is a  $\mathcal{C}^1$ -map.*

The second step is to solve equation (2.14). More precisely, for  $\varepsilon$  small enough we will find a point  $\xi$  in  $M$  such that equation (2.14) is satisfied. To this end we introduce the reduced energy function  $\tilde{I}_\varepsilon : M \rightarrow \mathbb{R}$  defined by

$$\tilde{I}_\varepsilon(\xi) := I_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}),$$

where  $I_\varepsilon$  is the variational functional defined in (2.4) whose critical points are the solutions to problem (2.5). It is easy to verify that  $\xi_\varepsilon$  is a critical point of  $\tilde{I}_\varepsilon$  if and only if the function  $u_\varepsilon = W_{\varepsilon,\xi_\varepsilon} + \phi_{\varepsilon,\xi_\varepsilon}$  is a critical point of  $I_\varepsilon$ .

In Lemmas 4.1 and 4.2 we compute the asymptotic expansion of the reduced functional  $\tilde{I}_\varepsilon$  with respect to the parameter  $\varepsilon$ . We prove the following result.

**Proposition 2.6.** *The expansion*

$$\tilde{I}_\varepsilon(\xi) = C \frac{c(\xi)^{\frac{n}{2}} a(\xi)^{\frac{p}{p-2} - \frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}} + o(1) = C\Gamma(\xi) + o(1),$$

*holds true  $\mathcal{C}^1$ -uniformly with respect to  $\xi$  as  $\varepsilon \rightarrow 0$ , where  $C = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^n} U^p dz$ .*

Using the previous propositions we now prove Theorem 1.3.

**Proof of Theorem 1.3.** Since  $K$  is a  $\mathcal{C}^1$ -stable critical set for  $\Gamma$ , by Proposition 2.6  $\tilde{I}_\varepsilon$  has a critical point  $\xi_\varepsilon \in M$  such that  $d_g(\xi_\varepsilon, K) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Hence,  $u_\varepsilon = W_{\varepsilon, \xi_\varepsilon} + \phi_{\varepsilon, \xi_\varepsilon}$  is a solution of (2.5), and the pair  $(u_\varepsilon, \Psi(u_\varepsilon))$  is a solution to the system (1.7) such that  $u_\varepsilon$  concentrates at a point  $\xi_0 \in K$  as  $\varepsilon \rightarrow 0$ .  $\square$

### 3. THE FINITE DIMENSIONAL REDUCTION

This section is devoted to the proof of Proposition 2.5. We denote by

$$(3.1) \quad \|u\|_g^2 := \int_M (|\nabla_g u|^2 + u^2) d\mu_g \quad \text{and} \quad |u|_{g,q}^q := \int_M |u|^q d\mu_g$$

the standard norms in the spaces  $H_g^1(M)$  and  $L^q(M)$ .

Equation (2.13) is equivalent to

$$(3.2) \quad L_{\varepsilon, \xi}(\phi) = N_{\varepsilon, \xi}(\phi) + S_{\varepsilon, \xi}(\phi) + R_{\varepsilon, \xi},$$

where

$$\begin{aligned} N_{\varepsilon, \xi}(\phi) &:= \Pi_{\varepsilon, \xi}^\perp \{i_\varepsilon^* [b(x) (f(W_{\varepsilon, \xi} + \phi) - f(W_{\varepsilon, \xi}) - f'(W_{\varepsilon, \xi})) \phi]\}, \\ S_{\varepsilon, \xi}(\phi) &:= \omega^2 \Pi_{\varepsilon, \xi}^\perp \{i_\varepsilon^* [b(x) (q^2 \Psi^2(W_{\varepsilon, \xi} + \phi) - 2q \Psi(W_{\varepsilon, \xi} + \phi)) (W_{\varepsilon, \xi} + \phi)]\}, \\ R_{\varepsilon, \xi} &:= \Pi_{\varepsilon, \xi}^\perp \{i_\varepsilon^* [b(x) f(W_{\varepsilon, \xi})] - W_{\varepsilon, \xi}\}. \end{aligned}$$

In order to solve equation (3.2) we will show that the operator  $T_{\varepsilon, \xi} : K_{\varepsilon, \xi}^\perp \rightarrow K_{\varepsilon, \xi}^\perp$  defined by

$$T_{\varepsilon, \xi}(\phi) := L_{\varepsilon, \xi}^{-1} (N_{\varepsilon, \xi}(\phi) + S_{\varepsilon, \xi}(\phi) + R_{\varepsilon, \xi})$$

has a fixed point. To this end we prove that  $T_{\varepsilon, \xi}$  is a contraction mapping on suitable ball in  $H_\varepsilon$ . We start with an estimate for  $R_{\varepsilon, \xi}$ .

**Lemma 3.1.** *There exist  $\varepsilon_0 > 0$  and  $C > 0$  such that, for any  $\xi \in M$  and any  $\varepsilon \in (0, \varepsilon_0)$ , the inequality*

$$\|R_{\varepsilon, \xi}\|_\varepsilon \leq C\varepsilon$$

*holds true.*

*Proof.* See Lemma 4.2 in [3].  $\square$

Next, we give an estimate for  $N_{\varepsilon, \xi}(\phi)$ .

**Lemma 3.2.** *There exist  $\varepsilon_0 > 0$ ,  $C > 0$  and  $\tilde{C} \in (0, 1)$  such that, for any  $\xi \in M$ ,  $\varepsilon \in (0, \varepsilon_0)$  and  $R > 0$ , the inequalities*

$$(3.3) \quad \|N_{\varepsilon, \xi}(\phi)\|_\varepsilon \leq C(\|\phi\|_\varepsilon^2 + \|\phi\|_\varepsilon^{p-1}),$$

$$(3.4) \quad \|N_{\varepsilon, \xi}(\phi_1) - N_{\varepsilon, \xi}(\phi_2)\|_\varepsilon \leq \tilde{C}\|\phi_1 - \phi_2\|_\varepsilon,$$

*hold true for  $\phi, \phi_1, \phi_2 \in \{\phi \in H_\varepsilon : \|\phi\|_\varepsilon \leq R\varepsilon\}$ .*

*Proof.* By direct computation we obtain

$$(3.5) \quad |f'(W_{\varepsilon, \xi} + v) - f'(W_{\varepsilon, \xi})| \leq \begin{cases} CW_{\varepsilon, \xi}^{p-3}|v| & 2 < p < 3, \\ C(W_{\varepsilon, \xi}^{p-3}|v| + |v|^{p-2}) & p \geq 3. \end{cases}$$

From the mean value theorem and inequality (2.11) we derive

$$\|N_{\varepsilon, \xi}(\phi_1) - N_{\varepsilon, \xi}(\phi_2)\|_\varepsilon \leq C|f'(W_{\varepsilon, \xi} + \phi_2 + t(\phi_1 - \phi_2)) - f'(W_{\varepsilon, \xi})|_{\frac{p}{p-2}, \varepsilon} \|\phi_1 - \phi_2\|_\varepsilon.$$

Using (3.5) we conclude that

$$C |f'(W_{\varepsilon,\xi} + \phi_2 + t(\phi_1 - \phi_2)) - f'(W_{\varepsilon,\xi})|_{\frac{p}{p-2},\varepsilon} < 1$$

provided  $\|\phi_1\|_\varepsilon$  and  $\|\phi_2\|_\varepsilon$  are small enough. The same estimates yield (3.3).  $\square$

Now we estimate  $S_{\varepsilon,\xi}(\phi)$ .

**Lemma 3.3.** *There exists  $\varepsilon_0 > 0$  and  $C > 0$  such that, for any  $\xi \in M$ ,  $\varepsilon \in (0, \varepsilon_0)$  and  $R > 0$ , the inequalities*

$$(3.6) \quad \|S_{\varepsilon,\xi}(\phi)\|_\varepsilon \leq C\varepsilon,$$

$$(3.7) \quad \|S_{\varepsilon,\xi}(\phi_1) - S_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \leq \ell_\varepsilon \|\phi_1 - \phi_2\|_\varepsilon,$$

hold true for  $\phi, \phi_1, \phi_2 \in \{\phi \in H_\varepsilon : \|\phi\|_\varepsilon \leq R\varepsilon\}$ , where  $\ell_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

*Proof.* Let us prove (3.6). From the definition of  $i^*$  and inequality (2.11) we derive

$$\begin{aligned} \|S_{\varepsilon,\xi}(\phi)\|_\varepsilon &\leq C \left( |\Psi^2(W_{\varepsilon,\xi} + \phi)(W_{\varepsilon,\xi} + \phi)|_{p',\varepsilon} + |\Psi(W_{\varepsilon,\xi} + \phi)(W_{\varepsilon,\xi} + \phi)|_{p',\varepsilon} \right) \\ &=: I_1 + I_2. \end{aligned}$$

For any  $t \in (2, \infty)$ , setting  $s := \frac{tp'}{t-p'}$  and  $\vartheta := \frac{2}{t} \in (1, 2)$  and applying Lemma 5.3 and Remark 5.2, we obtain

$$\begin{aligned} I_2 &\leq C \frac{1}{\varepsilon^{2/p'}} \left( \int_M |\Psi(W_{\varepsilon,\xi} + \phi)|^t d\mu_g \right)^{\frac{1}{t}} \left( \int_M |W_{\varepsilon,\xi} + \phi|^s d\mu_g \right)^{\frac{1}{s}} \\ &\leq C \frac{1}{\varepsilon^{2/p'}} \|\Psi(W_{\varepsilon,\xi} + \phi)\|_g \left( \varepsilon^{\frac{2}{s}} \left( \frac{1}{\varepsilon^2} \int_M |W_{\varepsilon,\xi}|^s d\mu_g \right)^{\frac{1}{s}} + |\phi|_{g,s} \right) \\ &\leq C \frac{1}{\varepsilon^{2/p'}} (\varepsilon^\vartheta + \|\phi\|_\varepsilon^2) \left( \varepsilon^{\frac{2}{s}} + \|\phi\|_\varepsilon \right) \\ &\leq C \left( \varepsilon^{\vartheta + \frac{2}{s} - \frac{2}{p'}} + \varepsilon^{\vartheta + 1 - \frac{2}{p'}} \right) = C \left( \varepsilon^{\vartheta - \frac{2}{t}} + \varepsilon^{\vartheta + 1 - \frac{2}{p'}} \right) \\ &\leq C\varepsilon \end{aligned}$$

for all  $\|\phi\|_\varepsilon \leq R\varepsilon$ . From this estimate we deduce that  $I_1 \leq C\varepsilon$  and, hence, (3.6) follows.

Next, we prove (3.7). From inequality (2.11) we obtain that

$$\begin{aligned} \|S_{\varepsilon,\xi}(\phi_1) - S_{\varepsilon,\xi}(\phi_2)\|_\varepsilon &\leq C \left[ |\Psi(W_{\varepsilon,\xi} + \phi_1) - \Psi(W_{\varepsilon,\xi} + \phi_2)| W_{\varepsilon,\xi} \right]_{p',\varepsilon} \\ &\quad + C \left[ |\Psi^2(W_{\varepsilon,\xi} + \phi_1) - \Psi^2(W_{\varepsilon,\xi} + \phi_2)| W_{\varepsilon,\xi} \right]_{p',\varepsilon} \\ &\quad + C |\Psi(W_{\varepsilon,\xi} + \phi_1)\phi_1 - \Psi(W_{\varepsilon,\xi} + \phi_2)\phi_2|_{p',\varepsilon} \\ &\quad + C |\Psi^2(W_{\varepsilon,\xi} + \phi_1)\phi_1 - \Psi^2(W_{\varepsilon,\xi} + \phi_2)\phi_2|_{p',\varepsilon} \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

By Remark 5.2 and Lemma 5.4 with  $s := \frac{3}{2}$ , for some  $\theta \in (0, 1)$  we have that

$$\begin{aligned} I_1^{p'} &\leq \frac{C}{\varepsilon^2} \left( \int_M |\Psi'(W_{\varepsilon,\xi} + \theta\phi_1 + (1-\theta)\phi_2)(\phi_1 - \phi_2)|^p \right)^{\frac{p'}{p}} \left( \frac{1}{\varepsilon^2} \int_M |W_{\varepsilon,\xi}|^{\frac{p'p}{p-p'}} \right)^{\frac{p-p'}{p}} \varepsilon^{\frac{2(p-p')}{p}} \\ &\leq C \frac{\varepsilon^{\frac{2(p-p')}{p}}}{\varepsilon^2} \left( \varepsilon^{\frac{4}{3}} + \|\phi_1\|_g + \|\phi_2\|_g \right)^{p'} \|\phi_1 - \phi_2\|_g^{p'} \\ &\leq Cl_\varepsilon \|\phi_1 - \phi_2\|_\varepsilon^{p'}, \end{aligned}$$

for  $\|\phi_1\|_\varepsilon, \|\phi_2\|_\varepsilon \leq R\varepsilon$ , with  $l_\varepsilon := \varepsilon^{\frac{p'(p-2)}{p}} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . From the estimate of  $I_1$ , recalling that  $0 \leq \Psi(u) \leq \frac{1}{q}$ , we derive

$$\begin{aligned} I_2^{p'} &= \frac{1}{\varepsilon^2} \int_M |\Psi(W_{\varepsilon,\xi} + \phi_1) + \Psi(W_{\varepsilon,\xi} + \phi_2)|^{p'} |\Psi(W_{\varepsilon,\xi} + \phi_1) - \Psi(W_{\varepsilon,\xi} + \phi_2)|^{p'} |W_{\varepsilon,\xi}|^{p'} \\ &\leq CI_1^{p'}. \end{aligned}$$

On the other hand, choosing  $\vartheta \in (1, 2)$  in Lemma 5.3 such that  $\vartheta p' > 2$  and applying Lemma 5.4 with  $s := \frac{3}{2}$ , we obtain

$$\begin{aligned} I_3^{p'} &\leq \frac{1}{\varepsilon^2} \int_M |\Psi'(W_{\varepsilon,\xi} + \theta\phi_1 + (1-\theta)\phi_2)(\phi_1 - \phi_2)|^{p'} |\phi_1|^{p'} \\ &\quad + \frac{1}{\varepsilon^2} \int_M |\Psi(W_{\varepsilon,\xi} + \phi_2)|^{p'} |\phi_1 - \phi_2|^{p'} \\ &\leq C \frac{1}{\varepsilon^2} \left( \int_M |\Psi'(W_{\varepsilon,\xi} + \theta\phi_1 + (1-\theta)\phi_2)(\phi_1 - \phi_2)|^p \right)^{\frac{p'}{p}} \left( \int_M |\phi_1|^{\frac{p'p}{p-p'}} \right)^{\frac{p-p'}{p}} \\ &\quad + C \frac{1}{\varepsilon^2} \left( \int_M |\phi_1 - \phi_2|^p \right)^{\frac{p'}{p}} \left( \int_M |\Psi(W_{\varepsilon,\xi} + \phi_2)|^{\frac{p'p}{p-p'}} \right)^{\frac{p-p'}{p}} \\ &\leq C \frac{1}{\varepsilon^2} \left( \varepsilon^{\frac{4}{3}} + \|\phi_1\|_g + \|\phi_2\|_g \right)^{p'} \|\phi_1 - \phi_2\|_g^{p'} \|\phi_1\|_g^{p'} \\ &\quad + C \frac{\varepsilon^{\vartheta p'}}{\varepsilon^2} (1 + \|\phi_2\|_\varepsilon^2) \|\phi_1 - \phi_2\|_g^{p'} \\ &\leq C \left( \frac{\varepsilon^{2p'}}{\varepsilon^2} + \frac{\varepsilon^{\vartheta p'}}{\varepsilon^2} \right) \|\phi_1 - \phi_2\|_\varepsilon^{p'} = l_\varepsilon \|\phi_1 - \phi_2\|_\varepsilon^{p'}, \end{aligned}$$

for  $\|\phi_1\|_\varepsilon, \|\phi_2\|_\varepsilon \leq R\varepsilon$ , where  $l_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Finally, from the estimate of  $I_2$  we derive  $I_4^{p'} \leq CI_3^{p'}$ . Collecting the previous estimates we obtain (3.7).  $\square$

**Proof of Proposition 2.5.** From Proposition 2.4 we deduce

$$\|T_{\varepsilon,\xi}(\phi)\|_\varepsilon \leq C (\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon + \|S_{\varepsilon,\xi}(\phi)\|_\varepsilon + \|R_{\varepsilon,\xi}\|_\varepsilon)$$

and

$$\|T_{\varepsilon,\xi}(\phi_1) - T_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \leq C \|N_{\varepsilon,\xi}(\phi_1) - N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon + C \|S_{\varepsilon,\xi}(\phi_1) - S_{\varepsilon,\xi}(\phi_2)\|_\varepsilon.$$

Lemmas 3.1, 3.2 and 3.3 imply that  $T_{\varepsilon,\xi}$  is a contraction in the ball centered at 0 of radius  $R\varepsilon$  in  $K_{\varepsilon,\xi}^\perp$ , for a suitable constant  $R$ . Hence,  $T_{\varepsilon,\xi}$  has a unique fixed point.

In order to prove that the map  $\xi \mapsto \phi_{\varepsilon,\xi}$  is  $\mathcal{C}^1$  we apply the implicit function theorem to the  $\mathcal{C}^1$ -function  $G : M \times H_\varepsilon \rightarrow H_\varepsilon$  defined by

$$\begin{aligned} G(\xi, u) &:= \Pi_{\varepsilon,\xi}^\perp \{ W_{\varepsilon,\xi} + \Pi_{\varepsilon,\xi}^\perp u - i_\varepsilon^* [b(x)f(W_{\varepsilon,\xi} + \Pi_{\varepsilon,\xi}^\perp u) + \omega^2 b(x)g(W_{\varepsilon,\xi} + \Pi_{\varepsilon,\xi}^\perp u)] \} \\ &\quad + \Pi_{\varepsilon,\xi} u. \end{aligned}$$

Note that  $G(\xi, \phi_{\varepsilon, \xi}) = 0$ . Next we show that the linearized operator  $\frac{\partial G}{\partial u}(\xi, \phi_{\varepsilon, \xi}) : H_\varepsilon \rightarrow H_\varepsilon$  defined by

$$\begin{aligned} & \frac{\partial G}{\partial u}(\xi, \phi_{\varepsilon, \xi})(u) \\ &= \Pi_{\varepsilon, \xi}^\perp \{ \Pi_{\varepsilon, \xi}^\perp(u) - i_\varepsilon^* [b(x)f'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^\perp(u) + \omega^2 b(x)g'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^\perp(u)] \} \\ & \quad + \Pi_{\varepsilon, \xi}(u) \end{aligned}$$

is invertible, provided  $\varepsilon$  is small enough. For any  $\phi$  with  $\|\phi\|_\varepsilon \leq C\varepsilon$  we have that

$$\begin{aligned} & \left\| \frac{\partial G}{\partial u}(\xi, \phi_{\varepsilon, \xi})(u) \right\|_\varepsilon \geq C \|\Pi_{\varepsilon, \xi}(u)\|_\varepsilon \\ & \quad + C \|\Pi_{\varepsilon, \xi}^\perp \{ \Pi_{\varepsilon, \xi}^\perp(u) - i_\varepsilon^* [f'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^\perp(u) + \omega^2 g'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^\perp(u)] \}\|_\varepsilon \\ & \geq C \|\Pi_{\varepsilon, \xi}(u)\|_\varepsilon + C \|L_{\varepsilon, \xi}(\Pi_{\varepsilon, \xi}^\perp(u))\|_\varepsilon \\ & \quad - C \|\Pi_{\varepsilon, \xi}^\perp \{ i_\varepsilon^* [(f'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) - f'(W_{\varepsilon, \xi})) \Pi_{\varepsilon, \xi}^\perp(u)] \}\|_\varepsilon \\ & \quad - C \|\Pi_{\varepsilon, \xi}^\perp \{ i_\varepsilon^* [\omega^2 g'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^\perp(u)] \}\|_\varepsilon \\ & \geq C \|\Pi_{\varepsilon, \xi}(u)\|_\varepsilon + C \|\Pi_{\varepsilon, \xi}^\perp(u)\|_\varepsilon - o(1) \|\Pi_{\varepsilon, \xi}^\perp(u)\|_\varepsilon \\ & \geq C \|u\|_\varepsilon. \end{aligned}$$

Indeed, by (3.5) we have

$$\begin{aligned} \|\Pi_{\varepsilon, \xi}^\perp \{ i_\varepsilon^* [(f'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) - f'(W_{\varepsilon, \xi})) \Pi_{\varepsilon, \xi}^\perp(u)] \}\|_\varepsilon & \leq C \left( \|\phi\|_\varepsilon^{p-2} + \|\phi\|_\varepsilon \right) \|\Pi_{\varepsilon, \xi}^\perp(u)\|_\varepsilon \\ & = o(1) \|\Pi_{\varepsilon, \xi}^\perp(u)\|_\varepsilon. \end{aligned}$$

Moreover,

$$\begin{aligned} & \|\Pi_{\varepsilon, \xi}^\perp \{ i_\varepsilon^* [\omega^2 g'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^\perp(u)] \}\|_\varepsilon \\ & \leq C |(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) (2q - 2q^2 \Psi(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi})) \Psi'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) [\Pi_{\varepsilon, \xi}^\perp(u)]|_{p', \varepsilon} \\ & \quad + C |[2q \Psi(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) - q^2 \Psi^2(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi})] \Pi_{\varepsilon, \xi}^\perp(u)|_{p', \varepsilon} \\ & := I_1 + I_2. \end{aligned}$$

From Lemma 5.4 we derive

$$\begin{aligned} I_1 & \leq \frac{C}{\varepsilon^{\frac{2}{p'}}} |W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}|_{g, 2} |\Psi'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^\perp(u)|_{g, \frac{4p'}{2-p'}} |2q - 2q^2 \Psi(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi})|_{g, \frac{4p'}{2-p'}} \\ & \leq C \frac{1}{\varepsilon^{\frac{2}{p'}}} \varepsilon(\varepsilon^{\frac{4}{3}} + \varepsilon) \|\Pi_{\varepsilon, \xi}^\perp u\|_g \leq \varepsilon^{2-\frac{2}{p'}} \|\Pi_{\varepsilon, \xi}^\perp u\|_g = o(1) \|\Pi_{\varepsilon, \xi}^\perp u\|_g, \end{aligned}$$

and, since  $0 \leq \Psi(u) \leq 1/q$ , from Lemma 5.3 with  $\vartheta p' > 2$  we get

$$\begin{aligned} I_2 & \leq \frac{C}{\varepsilon^{\frac{2}{p'}}} |\Pi_{\varepsilon, \xi}^\perp u|_{g, p} |\Psi(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi})|_{g, \frac{p'p}{p-p'}} \\ & \leq C \frac{\varepsilon^\vartheta}{\varepsilon^{\frac{2}{p'}}} \left( 1 + \|\phi_{\varepsilon, \xi}\|_\varepsilon^2 \right) \|\Pi_{\varepsilon, \xi}^\perp u\|_g = o(1) \|\Pi_{\varepsilon, \xi}^\perp u\|_g \end{aligned}$$

This concludes the proof.  $\square$

## 4. THE REDUCED ENERGY

This section is devoted to the proof of Proposition 2.6.

**Lemma 4.1.** *The following estimate*

$$(4.1) \quad \begin{aligned} \tilde{I}_\varepsilon(\xi) &= I_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \\ &= I_\varepsilon(W_{\varepsilon,\xi}) + o(1) = J_\varepsilon(W_{\varepsilon,\xi}) + \frac{\omega^2}{2} G_\varepsilon(W_{\varepsilon,\xi}) + o(1) \end{aligned}$$

holds true  $C^0$ -uniformly with respect to  $\xi$  as  $\varepsilon$  goes to zero. Moreover, setting  $\xi(y) := \exp_\xi(y)$ ,  $y \in B(0, r)$ , we have that

$$\begin{aligned} \left( \frac{\partial}{\partial y_h} \tilde{I}_\varepsilon(\xi(y)) \right) \Big|_{y=0} &= \left( \frac{\partial}{\partial y_h} I_\varepsilon(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \right) \Big|_{y=0} \\ &= \left( \frac{\partial}{\partial y_h} I_\varepsilon(W_{\varepsilon,\xi(y)}) \right) \Big|_{y=0} + o(1) \\ &= \left( \frac{\partial}{\partial y_h} J_\varepsilon(W_{\varepsilon,\xi(y)}) \right) \Big|_{y=0} + \frac{\omega^2}{2} \left( \frac{\partial}{\partial y_h} G_\varepsilon(W_{\varepsilon,\xi(y)}) \right) \Big|_{y=0} + o(1), \end{aligned}$$

$C^0$ -uniformly with respect to  $\xi$  as  $\varepsilon$  goes to zero.

*Proof.* In Lemma 5.1 of [3] we have proved the following two estimates:

$$J_\varepsilon(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - J_\varepsilon(W_{\varepsilon,\xi(y)}) = o(1),$$

$$(J'_\varepsilon(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - J'_\varepsilon(W_{\varepsilon,\xi(y)})) \left[ \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right] = o(1).$$

To complete the proof we shall prove the the following three estimates:

$$(4.2) \quad G_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - G_\varepsilon(W_{\varepsilon,\xi}) = o(1),$$

$$(4.3) \quad [G'_\varepsilon(W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) - G'_\varepsilon(W_{\varepsilon,\xi_0})] \left[ \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right] = o(1),$$

$$(4.4) \quad \left( J'_\varepsilon(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) + \frac{\omega^2}{2} G'_\varepsilon(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \right) \left[ \frac{\partial}{\partial y_h} \phi_{\varepsilon,\xi(y)} \right] = o(1).$$

We start with (4.2). For some  $\theta \in [0, 1]$  we have

$$\begin{aligned} &G_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - G_\varepsilon(W_{\varepsilon,\xi}) \\ &= \frac{1}{\varepsilon^2} \int_M b(x) \left[ \Psi(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})^2 - \Psi(W_{\varepsilon,\xi})(W_{\varepsilon,\xi})^2 \right] \\ &= \frac{1}{\varepsilon^2} \int_M b(x) \Psi'(W_{\varepsilon,\xi} + \theta \phi_{\varepsilon,\xi}) [\phi_{\varepsilon,\xi}] (W_{\varepsilon,\xi})^2 \\ &\quad + \frac{1}{\varepsilon^2} \int_M b(x) \Psi(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) [2\phi_{\varepsilon,\xi} W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}^2] \end{aligned}$$

Since  $\|\phi_{\varepsilon,\xi}\|_\varepsilon \leq C\varepsilon$ , from Lemma 5.4 we obtain (4.2).

Next, we prove (4.3). For some  $\theta \in [0, 1]$  we have

$$\begin{aligned}
& [G'_\varepsilon(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0}) - G'_\varepsilon(W_{\varepsilon, \xi_0})] \left[ \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \Big|_{y=0} \right] \\
& \leq \frac{q}{2\varepsilon^2} \left| \int_M b(x) \{ [2\Psi(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0}) - \Psi(W_{\varepsilon, \xi_0})] - [q\Psi^2(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0}) - q\Psi^2(W_{\varepsilon, \xi_0})] \} \right. \\
& \quad \cdot W_{\varepsilon, \xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \Big|_{y=0} \left. \right| \\
& \quad + \left| \frac{q}{2\varepsilon^2} \int_M 2b(x) [\Psi(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0}) - q\Psi^2(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0})] \phi_{\varepsilon, \xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \Big|_{y=0} \right| \\
& \leq \left| \frac{q}{2\varepsilon^2} \int_M 2b(x) \Psi'(W_{\varepsilon, \xi_0} + \theta\phi_{\varepsilon, \xi_0})(\phi_{\varepsilon, \xi_0}) W_{\varepsilon, \xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \Big|_{y=0} \right| \\
& \quad + \left| \frac{q}{\varepsilon^2} \int_M b(x) \Psi(W_{\varepsilon, \xi_0} + \theta\phi_{\varepsilon, \xi_0}) \Psi'(W_{\varepsilon, \xi_0} + \theta\phi_{\varepsilon, \xi_0})(\phi_{\varepsilon, \xi_0}) W_{\varepsilon, \xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \Big|_{y=0} \right| \\
& \quad + \left| \frac{q}{\varepsilon^2} \int_M b(x) \Psi(W_{\varepsilon, \xi_0}) \phi_{\varepsilon, \xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \Big|_{y=0} \right| \\
& \quad + \left| \frac{q}{\varepsilon^2} \int_M b(x) \Psi'(W_{\varepsilon, \xi_0} + \theta\phi_{\varepsilon, \xi_0})(\phi_{\varepsilon, \xi_0}) \phi_{\varepsilon, \xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \Big|_{y=0} \right| \\
& \quad + \left| \frac{q}{2\varepsilon^2} \int_M b(x) \Psi^2(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0})(\phi_{\varepsilon, \xi_0}) \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \Big|_{y=0} \right| \\
& := I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}$$

From Lemma 5.4, Remark 5.2 and equations (2.8), (2.9), (2.6), (2.7), recalling that  $\|\phi_{\varepsilon, \xi(y)}\|_\varepsilon \leq C\varepsilon$ , we get

$$\begin{aligned}
I_1 & \leq C \frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2} \left( \int_M [\Psi'(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0})(\phi_{\varepsilon, \xi_0})]^3 \right)^{\frac{1}{3}} \left( \frac{1}{\varepsilon^2} \int_M W_{\varepsilon, \xi_0}^3 \right)^{\frac{1}{3}} \left( \frac{1}{\varepsilon^2} \int_M \left[ \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \Big|_{y=0} \right]^3 \right)^{\frac{1}{3}} \\
& \leq C \varepsilon^{\frac{4}{3}} \left( \int_{\mathbb{R}^2} \left[ \sum_{k=1}^2 \left| \frac{1}{\varepsilon} \frac{\partial U}{\partial z_k}(z) \chi(\varepsilon z) + \left( \chi(\varepsilon z) + \frac{\partial \chi}{\partial z_k}(\varepsilon z) \right) U(z) \right|^3 \right] dz \right)^{\frac{1}{3}} \\
& \leq C \varepsilon^{\frac{4}{3}} \frac{1}{\varepsilon} = O(\varepsilon^{\frac{1}{3}})
\end{aligned}$$

In a similar way, using Lemma 5.4 and embedding the first and the second term in  $L^6$  and the third one in  $L^{3/2}$ , we get

$$I_4 \leq C \frac{1}{\varepsilon^2} [\varepsilon^{4/3} \|\phi_{\varepsilon, \xi}\|_\varepsilon + \|\phi_{\varepsilon, \xi}\|_\varepsilon^2] \|\phi_{\varepsilon, \xi}\|_\varepsilon \varepsilon^{\frac{4}{3}-1} = O(\varepsilon^{\frac{4}{3}}).$$

For  $I_3$  by Lemma 5.3 we have

$$\begin{aligned}
I_3 &\leq C \frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2} \left( \int_M [\Psi(W_{\varepsilon, \xi_0})]^3 \right)^{\frac{1}{3}} \left( \frac{1}{\varepsilon^2} \int_M \phi_{\varepsilon, \xi_0}^3 \right)^{\frac{1}{3}} \left( \frac{1}{\varepsilon^2} \int_M \left[ \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \Big|_{y=0} \right]^3 \right)^{\frac{1}{3}} \\
&\leq C \frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2} \|\Psi(W_{\varepsilon, \xi_0})\|_g \|\phi_{\varepsilon, \xi_0}\|_\varepsilon \left( \int_{\mathbb{R}^2} \left[ \sum_{k=1}^2 \left| \frac{1}{\varepsilon} \frac{\partial U}{\partial z_k}(z) \chi(\varepsilon z) + \left( \chi(\varepsilon z) + \frac{\partial \chi}{\partial z_k}(\varepsilon z) \right) U(z) \right|^2 \right] dz \right)^{\frac{1}{3}} \\
&\leq C \frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2} \varepsilon^{\frac{5}{3}} \varepsilon \frac{1}{\varepsilon} = O(\varepsilon)
\end{aligned}$$

and, from the estimate for  $I_3$ , since  $0 < \Psi(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0}) < 1/q$ , we obtain

$$I_5 \leq C I_3 = O(\varepsilon).$$

Finally, we prove (4.4). Following the proof of Lemma 5.1 in [3], we need only to prove that

$$\left| G'_\varepsilon(W_{\varepsilon, \xi(y)} + \phi_{\varepsilon, \xi(y)}) [Z_{\varepsilon, \xi(y)}^l] \right| = o(1),$$

that is

$$\left| \frac{1}{\varepsilon^2} \int_M [\Psi(W_{\varepsilon, \xi(y)} + \phi_{\varepsilon, \xi(y)}) - q \Psi^2(W_{\varepsilon, \xi(y)} + \phi_{\varepsilon, \xi(y)})] (W_{\varepsilon, \xi(y)} + \phi_{\varepsilon, \xi(y)}) Z_{\varepsilon, \xi(y)}^l \right| = o(1).$$

We have

$$\begin{aligned}
&\left| \frac{1}{\varepsilon^2} \int_M [\Psi(W_{\varepsilon, \xi(y)} + \phi_{\varepsilon, \xi(y)}) - q \Psi^2(W_{\varepsilon, \xi(y)} + \phi_{\varepsilon, \xi(y)})] (W_{\varepsilon, \xi(y)} + \phi_{\varepsilon, \xi(y)}) Z_{\varepsilon, \xi(y)}^l \right| \\
&\leq \frac{C}{\varepsilon^2} \int_M |\Psi(W_{\varepsilon, \xi(y)} + \phi_{\varepsilon, \xi(y)}) (W_{\varepsilon, \xi(y)} + \phi_{\varepsilon, \xi(y)}) Z_{\varepsilon, \xi(y)}^l| \\
&\quad + \frac{C}{\varepsilon^2} \int_M |\Psi^2(W_{\varepsilon, \xi(y)} + \phi_{\varepsilon, \xi(y)}) (W_{\varepsilon, \xi(y)} + \phi_{\varepsilon, \xi(y)}) Z_{\varepsilon, \xi(y)}^l| := I_1 + I_2.
\end{aligned}$$

By Proposition 2.3, we have that  $\|Z_{\varepsilon, \xi(y)}^l\|_\varepsilon = O(1)$ . So, by Lemma 5.3 and Remark 5.2, we have

$$\begin{aligned}
I_1 &\leq C \frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2} \left( \int_M [\Psi(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0})]^3 \right)^{\frac{1}{3}} \left( \frac{1}{\varepsilon^2} \int_M (W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0})^3 \right)^{\frac{1}{3}} \left( \frac{1}{\varepsilon^2} \int_M |Z_{\varepsilon, \xi(y)}^l|^3 \right)^{\frac{1}{3}} \\
&\leq C \frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2} \|\Psi(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0})\|_g (\|W_{\varepsilon, \xi_0}\|_{3, \varepsilon} + \|\phi_{\varepsilon, \xi_0}\|_\varepsilon) \|Z_{\varepsilon, \xi(y)}^l\|_\varepsilon = O(\varepsilon).
\end{aligned}$$

Again, as  $0 < \Psi(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0}) < 1/q$ , we obtain

$$I_2 \leq C I_1 = O(\varepsilon).$$

This concludes the proof.  $\square$

**Lemma 4.2.** *The expansion*

$$I_\varepsilon(W_{\varepsilon, \xi}) = \left( \frac{1}{2} - \frac{1}{p} \right) \frac{c(\xi)^{\frac{n}{2}} a(\xi)^{\frac{p}{p-2} - \frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}} \int_{\mathbb{R}^n} U^p dz + o(1)$$

holds true  $\mathcal{C}^1$ -uniformly with respect to  $\xi \in M$ .



*Proof.* In Lemma 5.2 of [3] we proved that

$$J_\varepsilon(W_{\varepsilon,\xi}) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{c(\xi)^{\frac{p}{2}} a(\xi)^{\frac{p}{p-2} - \frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}} \int_{\mathbb{R}^n} U^p dz + O(\varepsilon).$$

Hence, it suffices to show now that  $|G_\varepsilon(W_{\varepsilon,\xi})| = o(1)$ ,  $\mathcal{C}^1$ -uniformly with respect to  $\xi \in M$ .

Regarding the  $\mathcal{C}^0$ -convergence, by Remark 5.2 and Lemma 5.3, we have that

$$\begin{aligned} |G_\varepsilon(W_{\varepsilon,\xi})| &\leq \frac{C}{\varepsilon^2} \int_M \Psi(W_{\varepsilon,\xi}) W_{\varepsilon,\xi}^2 d\mu_g \\ &\leq C \frac{\varepsilon}{\varepsilon^2} \left( \int_M \Psi(W_{\varepsilon,\xi})^2 \right)^{\frac{1}{2}} \left( \frac{1}{\varepsilon^2} \int_M W_{\varepsilon,\xi}^4 \right)^{\frac{1}{2}} \\ &\leq C \frac{1}{\varepsilon} \|\Psi(W_{\varepsilon,\xi})\|_g \leq \frac{\varepsilon^{\frac{5}{3}}}{\varepsilon} = O(\varepsilon^{\frac{2}{3}}). \end{aligned}$$

Regarding the  $\mathcal{C}^1$ -convergence observe that

$$\begin{aligned} \left| \frac{\partial}{\partial y_h} G_\varepsilon(W_{\varepsilon,\xi}) \Big|_{y=0} \right| &\leq \left| \frac{C}{\varepsilon^2} \frac{\partial}{\partial y_h} \int_M \Psi(W_{\varepsilon,\xi(y)}) W_{\varepsilon,\xi(y)}^2 \Big|_{y=0} d\mu_g \right| \\ &\leq \left| \frac{C}{\varepsilon^2} \int_M \Psi(W_{\varepsilon,\xi(y)}) 2W_{\varepsilon,\xi(y)} \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} d\mu_g \right| \\ &\quad + \left| \frac{C}{\varepsilon^2} \int_M W_{\varepsilon,\xi(y)}^2 \Psi'(W_{\varepsilon,\xi(y)}) \left[ \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \Big|_{y=0} \right] d\mu_g \right| \\ &:= I_1 + I_2. \end{aligned}$$

Now, from Remark 5.2, Lemma 5.3, and the estimates (2.8) and (2.9), we derive

$$\begin{aligned} I_1 &\leq C \frac{\varepsilon^{\frac{8}{5}}}{\varepsilon^2} \left( \int_M \Psi(W_{\varepsilon,\xi(y)})^5 \right)^{\frac{1}{5}} \left( \frac{1}{\varepsilon^2} \int_M W_{\varepsilon,\xi(y)}^{\frac{5}{2}} \right)^{\frac{2}{5}} \left( \frac{1}{\varepsilon^2} \int_M \left( \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right)^{\frac{5}{2}} \right)^{\frac{2}{5}} \\ &\leq C \frac{\varepsilon^{\frac{8}{5}}}{\varepsilon^2} \varepsilon^{\frac{8}{5}} \frac{1}{\varepsilon} = o(1). \end{aligned}$$

On the other hand, from Remark 5.2, the proof of Lemma 5.4, and the estimates (2.8) and (2.9), for some  $t \in (1, 3/2)$  we obtain

$$\begin{aligned}
I_2 &\leq C \frac{\varepsilon^{\frac{2}{t}}}{\varepsilon^2} \left( \frac{1}{\varepsilon^2} \int_M W_{\varepsilon, \xi(h)}^{2t} \right)^{\frac{1}{t}} \left( \int_M \left( \Psi'(W_{\varepsilon, \xi(y)}) \left[ \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(h)} \Big|_{y=0} \right] \right)^{t'} \right)^{\frac{1}{t'}} \\
&\leq C \frac{\varepsilon^{\frac{2}{t}}}{\varepsilon^2} \left\| \Psi'(W_{\varepsilon, \xi(y)}) \left[ \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(h)} \Big|_{y=0} \right] \right\|_g \\
&\leq C \frac{\varepsilon^{\frac{2}{t}}}{\varepsilon^2} \varepsilon^{\frac{4}{3}} \left| \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(h)} \Big|_{y=0} \right|_{g,6} \\
&\leq C \frac{\varepsilon^{\frac{2}{t}}}{\varepsilon^2} \varepsilon^{\frac{4}{3}} \varepsilon^{\frac{1}{3}} \left( \frac{1}{\varepsilon^2} \int_M \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(h)} \Big|_{y=0} \right)^6 \right)^{\frac{1}{6}} \\
&\leq C \frac{\varepsilon^{\frac{2}{t}}}{\varepsilon^2} \varepsilon^{\frac{4}{3}} \varepsilon^{\frac{1}{3}} \frac{1}{\varepsilon} = C \varepsilon^{\frac{2}{t} - \frac{4}{3}} = o(1).
\end{aligned}$$

This concludes the proof.  $\square$

## 5. SOME ESTIMATES INVOLVING $\Psi$

We start by pointing out the following facts.

**Remark 5.1.** *There exists a constant  $C > 0$  such that, for every  $\varphi \in H_g^1(M)$  and every  $0 < \varepsilon < 1$ , we have*

$$\begin{aligned}
C \|\varphi\|_g^2 &= C \int_M (|\nabla_g \varphi|^2 + \varphi^2) d\mu_g \\
&\leq \int_M \left( c(x) |\nabla_g \varphi|^2 + \frac{d(x)}{\varepsilon^2} \varphi^2 \right) d\mu_g = \|\varphi\|_\varepsilon^2.
\end{aligned}$$

**Remark 5.2.** *The following estimates*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} |W_{\varepsilon, \xi}|_{g,p}^p \leq C |U|_p^p, \quad p \geq 2,$$

$$\lim_{\varepsilon \rightarrow 0} |\nabla_g W_{\varepsilon, \xi}|_{g,2}^2 \leq C |\nabla U|_2^2$$

*hold true uniformly with respect to  $\xi \in M$ .*

Abusing notation we write

$$\|u\|_g^2 = \int_M (c(x) |\nabla_g \varphi|^2 + b(x) u^2) d\mu_g.$$

This norm is equivalent to the standard norm (3.1) of  $H_g^1(M)$ . From equations (2.1), (2.2) and (2.3) we obtain

$$\begin{aligned}
(5.1) \quad \|\Psi(u)\|_g^2 &= \int_M b(x) q u^2 \Psi(u) d\mu_g - \int_M b(x) q^2 u^2 (\Psi(u))^2 d\mu_g \\
&\leq C \int_M u^2 \Psi(u) d\mu_g,
\end{aligned}$$

$$\begin{aligned}
(5.2) \quad \|\Psi'(u)[h]\|_g^2 &= \int_M 2b(x)qu(1-q\Psi(u))h\Psi'(u)[h] d\mu_g \\
&\quad - \int_M b(x)q^2u^2(\Psi'(u)[h])^2 d\mu_g \\
&\leq C \int_M |u||h||\Psi'(u)[h]| d\mu_g,
\end{aligned}$$

for all  $u, h \in H_g^1(M)$ .

**Lemma 5.3.** *Given  $\vartheta \in (1, 2)$  there is a constant  $C > 0$  such that the inequality*

$$\|\Psi(W_{\varepsilon, \xi} + \varphi)\|_g \leq C(\varepsilon^\vartheta + \|\varphi\|_g^2)$$

*holds true for every  $\varphi \in H_g^1(M)$ ,  $\xi \in M$  and small enough  $\varepsilon > 0$ .*

*Proof.* Let  $t \in (2, \infty)$  be such that  $\frac{2}{t'} = \vartheta$  where  $t'$  is the exponent conjugate to  $t$ . From inequality (5.1) we obtain

$$\begin{aligned}
\|\Psi(W_{\varepsilon, \xi} + \varphi)\|_g^2 &\leq C \left( \int_M [\Psi(W_{\varepsilon, \xi} + \varphi)]^t d\mu_g \right)^{1/t} \left( \int_M (W_{\varepsilon, \xi} + \varphi)^{2t'} d\mu_g \right)^{1/t'} \\
&\leq C \|\Psi(W_{\varepsilon, \xi} + \varphi)\|_g \|W_{\varepsilon, \xi} + \varphi\|_{g, 2t'}^2.
\end{aligned}$$

Thus, by Remark 5.2,

$$\begin{aligned}
\|\Psi(W_{\varepsilon, \xi} + \varphi)\|_g &\leq C \left( \varepsilon^{2/t'} \left( \frac{1}{\varepsilon^2} \int_M W_{\varepsilon, \xi}^{2t'} d\mu_g \right)^{1/t'} + \left( \int_M \varphi^{2t'} d\mu_g \right)^{1/t'} \right) \\
&\leq C(\varepsilon^\vartheta + \|\varphi\|_g^2),
\end{aligned}$$

as claimed.  $\square$

**Lemma 5.4.** *Given  $s \in (1, 2)$  there is a constant  $C > 0$  such that the inequality*

$$\|\Psi'(W_{\varepsilon, \xi} + k)[h]\|_g \leq C\|h\|_g \left( \varepsilon^{\frac{2}{s}} + \|k\|_g \right)$$

*holds true for every  $k, h \in H_g^1(M)$ ,  $\xi \in M$  and small enough  $\varepsilon > 0$ .*

*Proof.* From inequality (5.2) we obtain,

$$\begin{aligned}
\|\Psi'(W_{\varepsilon, \xi} + k)[h]\|_g^2 &\leq C \int_M |W_{\varepsilon, \xi} + k| |h| |\Psi'(W_{\varepsilon, \xi} + k)[h]| d\mu_g \\
&\leq C \left( \int_M |W_{\varepsilon, \xi}| |h| |\Psi'(W_{\varepsilon, \xi} + k)[h]| d\mu_g + \int_M |k| |h| |\Psi'(W_{\varepsilon, \xi} + k)[h]| d\mu_g \right) \\
&=: I_1 + I_2.
\end{aligned}$$

Set  $t := 2s' \in (4, \infty)$ , where  $s'$  is the conjugate exponent to  $s$ . Using Remark 5.2 we conclude that

$$\begin{aligned}
I_1 &\leq C |\Psi'(W_{\varepsilon, \xi} + k)[h]|_{g, t} |h|_{g, t} |W_{\varepsilon, \xi}|_{g, s} \\
&= C \|\Psi'(W_{\varepsilon, \xi} + k)[h]\|_g \|h\|_g \varepsilon^{\frac{2}{s}} \left( \frac{1}{\varepsilon^2} \int_M W_{\varepsilon, \xi}^s d\mu_g \right)^{1/s} \\
&= C \|\Psi'(W_{\varepsilon, \xi} + k)[h]\|_g \|h\|_g \varepsilon^{\frac{2}{s}}.
\end{aligned}$$

Since

$$I_2 \leq C |\Psi'(W_{\varepsilon, \xi} + k)[h]|_{g, 3} |h|_{g, 3} \|k\|_{g, 3} \leq C \|\Psi'(W_{\varepsilon, \xi} + k)[h]\|_g \|h\|_g \|k\|_g,$$

the claim follows.  $\square$

**Lemma 5.5.** *Consider the functions*

$$\tilde{v}_{\varepsilon,\xi}(z) := \begin{cases} \Psi(W_{\varepsilon,\xi})(\exp_{\xi}(\varepsilon z)) & \text{for } z \in B(0, r/\varepsilon), \\ 0 & \text{for } z \in \mathbb{R}^2 \setminus B(0, r/\varepsilon). \end{cases}$$

*Then, for any  $\vartheta \in (1, 2)$ , there exists a constant  $C > 0$ , independent of  $\varepsilon, \xi$ , such that*

$$\begin{aligned} |\tilde{v}_{\varepsilon,\xi}(z)|_{L^2(\mathbb{R}^3)} &\leq C\varepsilon^{\vartheta-1}, \\ |\nabla \tilde{v}_{\varepsilon,\xi}(z)|_{L^2(\mathbb{R}^3)} &\leq C\varepsilon^{\vartheta}. \end{aligned}$$

*Proof.* After a change of variables we have that

$$\begin{aligned} &\int_{B_g(\xi, r)} |\nabla \Psi(W_{\varepsilon,\xi})|^2 + |\Psi(W_{\varepsilon,\xi})|^2 d\mu_g \\ &= \varepsilon^2 \int_{B(0, r/\varepsilon)} |g_{\xi}(\varepsilon z)|^{1/2} \left( \sum_{ij} g_{\xi}^{ij}(\varepsilon z) \frac{1}{\varepsilon^2} \frac{\partial \tilde{v}_{\varepsilon,\xi}(z)}{\partial z_i} \frac{\partial \tilde{v}_{\varepsilon,\xi}(z)}{\partial z_i} + \tilde{v}_{\varepsilon,\xi}^2(z) \right) dz. \end{aligned}$$

Thus

$$\|\Psi(W_{\varepsilon,\xi})\|_g^2 \geq C(|\nabla \tilde{v}_{\varepsilon,\xi}|_{L^2(\mathbb{R}^3)}^2 + \varepsilon^2 |\tilde{v}_{\varepsilon,\xi}|_{L^2(\mathbb{R}^3)}^2).$$

This, combined with Lemma 5.3, gives

$$|\nabla \tilde{v}_{\varepsilon,\xi}|_{L^2(\mathbb{R}^3)} + \varepsilon |\tilde{v}_{\varepsilon,\xi}|_{L^2(\mathbb{R}^3)} \leq C\varepsilon^{\vartheta},$$

as claimed.  $\square$

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